Math 434 Assignment 5

Due May 24

Assignments will be collected in class.

1. Let λ be a limit ordinal, and for each $\alpha < \lambda$, let X_α be a set with $X_\alpha \subseteq X_\beta$ when $\alpha < \beta$. Let $X_\lambda = \bigcup_{\alpha<\lambda} X_\alpha$.

   Suppose that each X_α is well-ordered by $\leq_\alpha$, and suppose that

   (a) If $\alpha < \beta$, then $\leq_\beta$ extends $\leq_\alpha$.

   (b) If $\alpha < \beta$, $x \in X_\alpha$, and $y \in X_\beta - X_\alpha$, then $x <_\beta y$.

   Prove that X_λ is well-ordered by setting $x \leq_\lambda y$ if and only if there is $\alpha < \lambda$ with $x \leq_\alpha y$.

2. The Σ_1 formulas are defined by:

   - if φ is Δ_0, then φ is Σ_1;
   - if $\varphi(x)$ is Σ_1, then $\exists x \varphi(x)$ is Σ_1;
   - if $\varphi(x)$ is Σ_1, then $\exists x \in y\, \varphi(x)$ and $\forall x \in y\, \varphi(x)$ are Σ_1.

   Similarly, the Π_1 formulas are defined by:

   - if φ is Δ_0, then φ is Π_1;
   - if $\varphi(x)$ is Π_1, then $\forall x \varphi(x)$ is Π_1;
   - $\varphi(x)$ is Π_1, then $\exists x \in y\, \varphi(x)$ and $\forall x \in y\, \varphi(x)$ are Π_1.

   (a) Let $M \models ZF$ be a transitive model of set theory. Given a set a, prove that

   $$M \models (\forall x \in y)\, \exists z\, \varphi(x, z)$$

   if and only if

   $$M \models \exists u\, (\forall x \in y)(\exists z \in u)\, \varphi(x, z)$$

   Hint: Use the Axiom Schema of Collection, which say that if for all $x \in A$ there exists a y with $\varphi(x, y)$, then there exists a set B such that for all $x \in A$ there is a $y \in B$ with $\varphi(x, y)$. This is a consequence of ZF.

   Let $M \subseteq N$ be transitive models of ZF, $\varphi(\bar{x})$ a formula, and $\bar{a} \in M$. Prove:

(b) If $\varphi(\bar{x})$ is a Σ_1 formula, then $M \vDash \varphi(\bar{a})$ implies that $N \vDash \varphi(\bar{a})$.

(c) If $\varphi(\bar{x})$ is a Π_1 formula, then $N \vDash \varphi(\bar{a})$ implies that $M \vDash \varphi(\bar{a})$.

3. For this question, work in ZFC+GCH. So for any cardinal κ, $|\mathcal{P}(\kappa)| = 2^\kappa = \kappa^+$, the least cardinal greater than κ. Let κ, μ be two infinite cardinals.

   (a) Prove that $\kappa^\kappa = \kappa^+$.

   (b) Prove that if $\mu^+ \geq \kappa$, then $\kappa^\mu = \mu^+$.

   (c) Prove that if $\mu^+ < \kappa$ and $\mu \geq cof(\kappa)$, then $\kappa^\mu = \kappa^+$. *Hint: κ must be singular and hence a limit cardinal (the proof of this fact might give you some inspiration). So if $\lambda < \kappa$, then $\lambda^+ < \kappa$. Also, a function $\kappa \to 2$ is determined by its restriction to each $\lambda < \kappa$.*

   Now assume that $\mu^+ < \kappa$ and $\mu < cof(\kappa)$. Then $\kappa > \aleph_0$, so either $\kappa = \lambda^+$ for some infinite cardinal λ, or κ is a limit cardinal.

   (d) Prove that if κ is a successor cardinal, then $\kappa^\mu = \kappa$.

   (e) Suppose that κ is a limit cardinal. Given a function $f \colon \mu \to \kappa$, recall that since $\mu < cof(\kappa)$, there is an infinite cardinal $\lambda < \kappa$ such that $\operatorname{ran}(f) \subseteq \lambda$.
   Prove that $\kappa^\mu = \sup_{\lambda<\kappa} \lambda^\mu$, so that $\kappa^\mu = \kappa$.

   We have shown that, assuming GCH,

$$\kappa^\mu = \begin{cases} \kappa & \text{if } \mu^+ < \kappa \text{ and } \mu < cof(\kappa) \\ \kappa^+ & \text{if } \mu^+ < \kappa \text{ and } \mu \geq cof(\kappa) \\ \mu^+ & \text{if } \mu^+ \geq \kappa \end{cases}.$$

4. Let X be a set. Prove that there is $Y \in \mathcal{P}(X) - \mathcal{P}_{def}(X)$.

5. Let Γ be a finite subset of ZFC. Prove that $ZFC \vdash Con(\Gamma)$. Conclude that ZFC is not finitely axiomatizable.